

ℓ -ADIC CLASS FIELD THEORY FOR REGULAR LOCAL RINGS

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ABSTRACT. In this paper, we prove the ℓ -adic abelian class field theory for henselian regular local rings of equi-characteristic assuming the surjectivity of Galois symbol maps, which is a ℓ -adic variant of a result of Matsumi [13].

1. INTRODUCTION

Let p be a prime number, and let A be an excellent henselian regular local ring over \mathbb{F}_p of dimension n with finite residue field (e.g., $\mathbb{F}_p[[X_1, \dots, X_n]]$). Let s be the closed point of $X := \text{Spec}(A)$ and let D be an effective divisor on X . By standard arguments in the higher class field theory (cf. [10], [19]), there is a localized Chern class map

$$(1.1) \quad c_{X,D,\ell^r} : H_s^n(X_{\text{Nis}}, \mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D)) / \ell^r \longrightarrow H_s^{2n}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes n})$$

for a prime number $\ell \neq p$ and $r > 0$. Here j denotes the natural open immersion $X - \text{Supp}(D) \hookrightarrow X$, $\mathcal{I}_D \subset \mathcal{O}_X$ denotes the defining ideal of D and $\mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D)$ denotes a certain Milnor K -sheaf in the Nisnevich topology (see Notation below). We will review the construction of this map in §3. The group $H_s^n(X_{\text{Nis}}, \mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D))$ plays the role of the K_n^M -idèle class group of $K := \text{Frac}(A)$ with modulus D . The main result of this paper is the bijectivity of this map, which is conditional with respect to the surjectivity of Galois symbol maps:

Theorem 1.2. *Assume $\mathbf{S}(K, n, \ell)$ holds (see Notation below for the precise statement of this condition). Then c_{X,D,ℓ^r} is bijective.*

A key to the proof of this theorem is the exactness of the cousin complex

$$0 \rightarrow H^{n+1}(\text{Spec}(K)_{\text{ét}}, \mu_{\ell^r}^{\otimes n}) \rightarrow \bigoplus_{y \in X^1} H_y^{n+2}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes n}) \rightarrow \bigoplus_{y \in X^2} H_y^{n+3}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes n}) \rightarrow \cdots,$$

which we prove using a result of Panin [17], Theorem 5.2 (cf. [2], [11]). In fact, we will prove this exactness for the spectrum of an arbitrary henselian local ring which is essentially étale over A (see Lemma 4.2 below). Using this local-global principle, we will deduce the bijectivity of c_{X,D,ℓ^r} from the bijectivity of Galois symbol maps for n -dimensional local fields [9], p. 672, Lemma 14 (1) (see §5 below for details).

We explain how Theorem 1.2 is related to the class field theory of K . For $D \subset X$ as before, there is a natural pairing of \mathbb{Z}/ℓ^r -modules

$$(1.3) \quad H_s^{2n}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes n}) \times H^1(U_{\text{ét}}, \mathbb{Z}/\ell^r) \longrightarrow H_s^{2n+1}(X_{\text{ét}}, \mu_{\ell^r}^{\otimes n}) \xrightarrow{\sim} \mathbb{Z}/\ell^r.$$

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Here U denotes $X - \text{Supp}(D)$, and the trace isomorphism is given by that of the closed point s and the cohomological purity [5], remark after 7.1.7 (cf. [6], XIX). This pairing together with c_{X,D,ℓ^r} induces a map

$$\rho_{X,D,\ell^r} : H_s^n(X_{\text{Nis}}, \mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D)) / \ell^r \longrightarrow \pi_1^{\text{ab}}(U) / \ell^r.$$

Taking projective limits with respect to D and r , we obtain the ℓ -adic reciprocity map of the class field theory of K :

$$\rho_{K,\ell\text{-adic}} : C_K := \varprojlim_{D \subset X} H_s^n(X_{\text{Nis}}, \mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D)) \longrightarrow G_K^{(\ell)},$$

where D runs through all effective divisors on X , and $G_K^{(\ell)}$ denotes the maximal pro- ℓ quotient of $\text{Gal}(K^{\text{ab}}/K)$ with K^{ab} the maximal abelian extension of K .

Corollary 1.4. *Assume $\mathbf{S}(K, n, \ell)$ holds. Then $\rho_{K,\ell\text{-adic}}$ has dense image with respect to the Krull topology on the Galois group, and its kernel consists of elements which are divisible in C_K by any power of ℓ .*

Indeed, by a theorem of Gabber [8], Théorème 5.4 (see also [20], Théorème 0.2, [7], I), the pairing (1.3) is a non-degenerate pairing of finite groups. This fact and Theorem 1.2 imply that the map ρ_{X,D,ℓ^r} is bijective for any D and r . Hence $\rho_{K,\ell\text{-adic}}$ has dense image and $\ker(\rho_{K,\ell\text{-adic}})$ agrees with $\bigcap_{r>0} \ell^r \cdot C_K$.

Corollary 1.4 extends results of Saito [21] and Matsumi [12] to higher-dimensional cases, and its p -adic variant is due to Matsumi [13].

NOTATION

For a commutative ring R with unity and an integer $q \geq 0$, $K_q^M(R)$ denotes the Milnor K -group of degree q defined as $(R^\times)^{\otimes q} / J$, where J denotes the subgroup of $(R^\times)^{\otimes q}$ generated by elements of the form $x_1 \otimes \cdots \otimes x_q$ with $x_i + x_j = 0$ or 1 for some $1 \leq i < j \leq q$. For a local ring R , R^h denotes the henselization of R . For a field F and a positive integer m prime to $\text{ch}(F)$, let $h_{F,m}^q$ be Galois symbol map due to Tate [23]

$$h_{F,m}^q : K_q^M(F) / m \longrightarrow H_{\text{Gal}}^q(F, \mu_m^{\otimes q}),$$

where the right hand side is the Galois cohomology of degree q of the absolute Galois group of F . In [1], §5, Bloch and Kato conjectured that this map is always bijective. For a prime number $\ell \neq \text{ch}(F)$, we define the condition $\mathbf{S}(F, q, \ell)$ as follows:

$\mathbf{S}(F, q, \ell) : h_{F',\ell}^q$ is surjective for any finite separable extension F'/F .

Remark 1.5. $h_{F,m}^q$ is known to be bijective in the following cases:

- 1) $q = 0$, $q = 1$ (Hilbert's theorem 90), $q = 2$ (Merkur'ev-Suslin [14]).
- 2) F is a finite field. Indeed, both $K_q^M(F)$ and $H_{\text{Gal}}^q(F, \mu_m^{\otimes q})$ are zero for $q \geq 2$.
- 3) $\text{ch}(F) \neq 2$ and $m = 2^r$ (Merkur'ev-Suslin [15], Voevodsky [24]).
- 4) F is a henselian discrete valuation ring of characteristic zero with residue field of characteristic $p > 0$ and $m = p^r$ (Bloch-Kato [1]).
- 5) The case that F is a henselian discrete valuation ring with residue field k , that m is prime to $\text{ch}(k)$ and that $h_{k,m}^q$ and $h_{k,m}^{q-1}$ are bijective ([9], p. 672, Lemma 14 (1)). In particular, in view of (2) and (4), $h_{F,m}^q$ is bijective for any local field

F in the sense of *loc. cit.*, and any m prime to $\text{ch}(F)$. The case $\text{ch}(F) > 0$ of this fact will be useful later.

For a scheme X , $\mathcal{K}_q^M(\mathcal{O}_X)$ denotes the Milnor K -sheaf of degree q in the Nisnevich topology on X . For an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, we define

$$\mathcal{K}_q^M(\mathcal{O}_X, \mathcal{I}) := \ker(\mathcal{K}_q^M(\mathcal{O}_X) \rightarrow \mathcal{K}_q^M(\mathcal{O}_X/\mathcal{I}))$$

For $q \geq 0$, X^q denotes the set of all points on X of codimension q . For a point $x \in X$, $\kappa(x)$ denotes the residue field of the local ring $\mathcal{O}_{X,x}$.

2. PRELIMINARY ON HENSELIAN DISCRETE VALUATION FIELDS

Let F be a henselian discrete valuation field and let O_F be its integer ring. Let π be a prime element of O_F . Put $X := \text{Spec}(O_F)$ and denote the generic point (resp. closed point) of X by η (resp. x). Let $U_F^1 = \{1 + \pi a \mid a \in O_F\}$ be the first unit group of O_F , and let $U^1 K_q^M(F)$ be the subgroup of $K_q^M(F)$ generated by the image of $U_F^1 \otimes (F^\times)^{\otimes q-1}$. The following elementary facts will be useful later:

Lemma 2.1. *Let m be a positive integer invertible in O_F .*

- (1) *Let I be a non-zero ideal of O_F with $I \neq O_F$, and let $K_q^M(O_F, I)$ be the kernel of the natural map $K_q^M(O_F) \rightarrow K_q^M(O_F/I)$. Then we have*

$$\text{Im}(K_q^M(O_F, I) \rightarrow K_q^M(F)) \subset U^1 K_q^M(F) \subset m \cdot K_q^M(F).$$

- (2) *The connecting map $H^q(\eta_{\text{ét}}, \mu_m^{\otimes q}) \rightarrow H_x^{q+1}(X_{\text{ét}}, \mu_m^{\otimes q})$ in the localization theory of étale cohomology is surjective for any $q \geq 1$.*

Proof. (1) The second inclusion follows from the fact that $U_F^1 \subset (O_F^\times)^m$, which is a direct consequence of Hensel's lemma. We show the first inclusion. Let k be the residue field of O_F . There is a short exact sequence (cf. [9], p. 616, Lemma 6)

$$0 \longrightarrow U^1 K_q^M(F) \longrightarrow K_q^M(F) \xrightarrow{\partial} K_q^M(k) \oplus K_{q-1}^M(k) \longrightarrow 0,$$

where the arrow ∂ is defined by the assignment

$$\begin{aligned} \{\pi, a_1, \dots, a_{q-1}\} &\mapsto \left(0, \{\overline{a_1}, \dots, \overline{a_{q-1}}\} \right) \\ \{a_1, \dots, a_q\} &\mapsto \left(\{\overline{a_1}, \dots, \overline{a_q}\}, 0 \right) \end{aligned}$$

with each $a_i \in O_F^\times$ and $\overline{a_i}$ denotes the residue class of a_i in k^\times . The assertion follows from this exact sequence and the assumption that $I \neq O_F$.

- (2) There is a natural map

$$H^{q-1}(x_{\text{ét}}, \mu_m^{\otimes q-1}) \longrightarrow H_x^{q+1}(X_{\text{ét}}, \mu_m^{\otimes q})$$

sending $\alpha \in H^{q-1}(x_{\text{ét}}, \mu_m^{\otimes q-1})$ to $\text{cl}_X(x) \cup \alpha$, where $\text{cl}_X(x) \in H_x^2(X_{\text{ét}}, \mu_m)$ denotes the cycle class of x ([4], Cycle, §2.1). This map is bijective by the purity for discrete valuation rings ([7], I.5). Let $\alpha \in H^{q-1}(x_{\text{ét}}, \mu_m^{\otimes q-1})$ be an arbitrary cohomology class. Since X is henselian local with closed point x , there is a unique $\alpha' \in H^{q-1}(X_{\text{ét}}, \mu_m^{\otimes q-1})$ that lifts α . By [4], Cycle, 2.1.3, the cohomology class $\{\pi\} \cup \alpha' \in H^q(\eta_{\text{ét}}, \mu_m^{\otimes q})$ maps to α under the composite map

$$H^q(\eta_{\text{ét}}, \mu_m^{\otimes q}) \longrightarrow H_x^{q+1}(X_{\text{ét}}, \mu_m^{\otimes q}) \xleftarrow{\sim} H^{q-1}(x_{\text{ét}}, \mu_m^{\otimes q-1}),$$

up to a sign, which implies the assertion. \square

3. LOCALIZED CHERN CLASS MAPS

In this section, we construct localized Chern class maps. Our construction is essentially the same as those in [10], [19]. We work under the following setting. Let X be a noetherian integral normal scheme. Let D be an effective Weil divisor on X . Let j be the open immersion $X - \text{Supp}(D) \hookrightarrow X$ and let $\mathcal{I}_D \subset \mathcal{O}_X$ be the defining ideal of D . Let ℓ be a prime number invertible on X , and let r be a positive integer. For $q \geq 0$ and $x \in X^a$, we define the localized Chern class map

$$(3.1) \quad \text{cl}_{X,D,x,\ell^r}^{q,\text{loc}} : H_x^a(X_{\text{Nis}}, \mathcal{K}_q^M(\mathcal{O}_X, \mathcal{I}_D)) / \ell^r \longrightarrow H_x^{q+a}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes q})$$

by induction on $a \geq 0$. The map c_{X,D,ℓ^r} in (1.1) is defined as $\text{cl}_{X,D,s,\ell^r}^{n,\text{loc}}$. We first recall the following general facts.

Lemma 3.2. *Let Z be a scheme of dimension d , and let \mathcal{F} be an abelian sheaf on Z_{Nis} . Then:*

- 1) *We have $H^t(Z_{\text{Nis}}, \mathcal{F}) = 0$ for any $t > d$.*
- 2) *For $z \in Z$, $H_z^t(Z_{\text{Nis}}, \mathcal{F})$ is isomorphic to*

$$\begin{cases} \text{Coker}(\mathcal{F}_z \rightarrow H^0(U_{\text{Nis}}, \mathcal{F})) & \text{if } t = 1 \\ H^{t-1}(U_{\text{Nis}}, \mathcal{F}) & \text{if } t \geq 2 \\ 0 & \text{if } t \geq \text{codim}_Z(z) + 1, \end{cases}$$

where U denotes $\text{Spec}(\mathcal{O}_{Z,z}^h) - \{z\}$.

- 3) *There is an exact sequence*

$$\bigoplus_{z \in Z^{d-1}} H_z^{d-1}(Z_{\text{Nis}}, \mathcal{F}) \longrightarrow \bigoplus_{z \in Z^d} H_z^d(Z_{\text{Nis}}, \mathcal{F}) \longrightarrow H^d(Z_{\text{Nis}}, \mathcal{F}) \longrightarrow 0.$$

Proof. For (1), see [19], Lemma 1.22. The assertion (2) follows from (1) and a standard localization argument using the excision in the Nisnevich topology. To show (3), we compute the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{z \in Z^u} H_z^{u+v}(Z_{\text{Nis}}, \mathcal{F}) \implies H^{u+v}(Z_{\text{Nis}}, \mathcal{F}).$$

We have $E_1^{u,v} = 0$ unless $0 \leq u \leq d$, and $E_1^{u,v} = 0$ for $v > 0$ by (2). Hence we have $E_2^{d,0} \simeq H^d(Z_{\text{Nis}}, \mathcal{F})$, which implies (3). \square

We construct the map (3.1) in three steps. Replacing X by X_x^h , the henselization of X at x if necessary, we assume that X is henselian local with closed point x .

Step 0. Assume $a = 0$. Then we have $X = x$ and D is zero. We define $\text{cl}_{X,0,x,\ell^r}^{q,\text{loc}}$ as the Galois symbol map $h_{x,\ell^r}^q : K_q^M(x) / \ell^r \rightarrow H^q(x_{\text{ét}}, \mu_{\ell^r}^{\otimes q})$.

Step 1. Assume $a = 1$. Then X is the spectrum of a henselian discrete valuation ring O_F with fraction field F . Let $\eta = \text{Spec}(F)$ be the generic point of X . We have $\text{Supp}(D) = x$, or otherwise, D is zero. Put $\mathcal{K}_D := \mathcal{K}_q^M(\mathcal{O}_X, \mathcal{I}_D)$ and $\mu_D := j_! \mu_{\ell^r}^{\otimes q}$ for

simplicity, which mean $\mathcal{K}_q^M(\mathcal{O}_X)$ and $\mu_{\ell^r}^{\otimes q}$, respectively, if $D = 0$ (i.e., $\mathcal{I}_D = \mathcal{O}_X$). We define $\text{cl}_{X,D,x,\ell^r}^{m,\text{loc}}$ as the map induced by the commutative diagram with exact rows

$$(3.3) \quad \begin{array}{ccccccc} H^0(X_{\text{Nis}}, \mathcal{K}_D)/\ell^r & \longrightarrow & K_q^M(F)/\ell^r & \longrightarrow & H_x^1(X_{\text{Nis}}, \mathcal{K}_D)/\ell^r & \longrightarrow & 0 \\ \downarrow & & h_{\eta,\ell^r}^q \downarrow & & & & \\ H^q(X_{\text{ét}}, \mu_D) & \longrightarrow & H^q(\eta_{\text{ét}}, \mu_{\ell^r}^{\otimes q}) & \xrightarrow{\delta} & H_x^{q+1}(X_{\text{ét}}, \mu_D) & \longrightarrow & 0. \end{array}$$

Here the upper row (resp. lower row) arises from a localization long exact sequence of Nisnevich (resp. étale) cohomology groups, and the upper row is exact by Lemma 3.2 (1). The arrow δ is surjective by Lemma 2.1 (2) (resp. the fact that $\mu_D|_x = 0$) if $D = 0$ (resp. $D \neq 0$). The left vertical arrow is the Galois symbol map $K_q^M(\mathcal{O}_F)/\ell^r \rightarrow H^q(X_{\text{ét}}, \mu_{\ell^r}^{\otimes q})$ (resp. zero map) if $D = 0$ (resp. $D \neq 0$). The left square commutes obviously if $D = 0$. If $D \neq 0$, then the top left arrow is zero by Lemma 2.1 (1), and the left square commutes. Thus we obtain the map $\text{cl}_{X,D,x,\ell^r}^{q,\text{loc}}$ in the case $a = 1$.

Step 2. Assume $a \geq 2$ and that the localized Chern class maps have been defined for points of codimension $\leq a - 1$. Put $\mathcal{K}_D := \mathcal{K}_q^M(\mathcal{O}_X, \mathcal{I}_D)$ and $\mu_D := j_! \mu_{\ell^r}^{\otimes q}$ as before. We define $\text{cl}_{X,D,x,\ell^r}^{q,\text{loc}}$ to be the map induced by the commutative diagram

$$(3.4) \quad \begin{array}{ccccc} \bigoplus_{y \in X^{a-2}} H_y^{a-2}(X_{\text{Nis}}, \mathcal{K}_D)/\ell^r & \rightarrow & \bigoplus_{y \in X^{a-1}} H_y^{a-1}(X_{\text{Nis}}, \mathcal{K}_D)/\ell^r & \rightarrow & H_x^a(X_{\text{Nis}}, \mathcal{K}_D)/\ell^r \rightarrow 0 \\ \bigoplus \text{cl}_{X,D,y,\ell^r}^{q,\text{loc}} \downarrow & & \bigoplus \text{cl}_{X,D,y,\ell^r}^{q,\text{loc}} \downarrow & & \\ \bigoplus_{y \in X^{a-2}} H_y^{q+a-2}(X_{\text{ét}}, \mu_D) & \rightarrow & \bigoplus_{y \in X^{a-1}} H_y^{q+a-1}(X_{\text{ét}}, \mu_D) & \rightarrow & H_x^{q+a}(X_{\text{ét}}, \mu_D), \end{array}$$

where the upper row is exact by Lemma 3.2 (3) for $X - \{x\}$ and the lower row is a complex. The left square commutes by the construction of the vertical arrows (cf. (3.3)). This completes the construction of the localized Chern class map (3.1).

Remark 3.5. One can check that the group $H_x^a(X_{\text{Nis}}, \mathcal{K}_q^M(\mathcal{O}_X, \mathcal{I}_D))/\ell^r$ depends only on $X - \text{Supp}(D)$ by repeating the arguments in this section.

4. KEY RESULTS ON ÉTALE COHOMOLOGY

Let A be as in the introduction and let ℓ be a prime number invertible in A . In this section we prove Lemma 4.2 below (compare with Lemma 3.2) using the following key fact due to Gabber ([8], Corollaire 4.3, [16], Théorème 5.1, [18], Théorème 4):

Theorem 4.1 (Gabber). *Let R' be an integral excellent henselian local ring of dimension d with residue field k , and let K' be the fraction field of R' . Then we have $\text{cd}_\ell(K') = d + \text{cd}_\ell(k)$ for any prime number $\ell \neq \text{ch}(k)$.*

Lemma 4.2. *Let R be a henselian local ring which is essentially étale over A (R may be A itself). Put $X := \text{Spec}(R)$ and $a := \dim(X)$. Then:*

- 1) For $y \in X$, we have $\text{cd}_\ell(y) \leq n + 1 - \text{codim}_X(y)$.

- 2) For an ℓ -primary torsion sheaf \mathcal{F} on $X_{\text{ét}}$ and $y \in X^c$ ($c \geq 0$), $H_y^t(X_{\text{ét}}, \mathcal{F})$ is isomorphic to

$$\begin{cases} \text{Coker}(H^{t-1}(\text{Spec}(\mathcal{O}_{X,y}^h)_{\text{ét}}, \mathcal{F}) \rightarrow H^{t-1}(U_{\text{ét}}, \mathcal{F})) & \text{if } t = n + 2 - c \\ H^{t-1}(U_{\text{ét}}, \mathcal{F}) & \text{if } t \geq n + 3 - c \\ 0 & \text{if } t \geq n + 2 + c, \end{cases}$$

where U denotes $\text{Spec}(\mathcal{O}_{X,y}^h) - \{y\}$.

- 3) Assume $a \geq 1$. Let x (resp. η) be the closed point (the generic point) of X . Let D be an effective divisor on X . Let j be the open immersion $X - \text{Supp}(D) \hookrightarrow X$ and put $\mu_D := j_! \mu_{\ell^r}^{\otimes n}$. Then the sequence

$$\begin{aligned} 0 \rightarrow H^{n+1}(\eta_{\text{ét}}, \mu_{\ell^r}^{\otimes n}) \rightarrow \bigoplus_{y \in X^1} H_y^{n+2}(X_{\text{ét}}, \mu_D) \rightarrow \bigoplus_{y \in X^2} H_y^{n+3}(X_{\text{ét}}, \mu_D) \rightarrow \\ \dots \rightarrow H_x^{n+a+1}(X_{\text{ét}}, \mu_D) \rightarrow 0 \end{aligned}$$

is exact.

Admitting this lemma, we first prove the following two consequences, where the notation remains as in Lemma 4.2 and K denotes $\text{Frac}(A)$.

Corollary 4.3. *If $a \geq 2$, then the sequence*

$$\bigoplus_{y \in X^{a-2}} H_y^{n+a-2}(X_{\text{ét}}, \mu_D) \rightarrow \bigoplus_{y \in X^{a-1}} H_y^{n+a-1}(X_{\text{ét}}, \mu_D) \rightarrow H_x^{n+a}(X_{\text{ét}}, \mu_D) \rightarrow 0$$

is exact.

Proof. Put $Y := X - \{x\}$. Since $n + 3 - a < n + a$, we have $H^{n+a-1}(Y_{\text{ét}}, \mu_D) \simeq H_x^{n+a}(X_{\text{ét}}, \mu_D)$ by Lemma 4.2 (2). We prove the sequence

$$(4.4) \quad \bigoplus_{y \in Y^{a-2}} H_y^{n+a-2}(Y_{\text{ét}}, \mu_D) \rightarrow \bigoplus_{y \in Y^{a-1}} H_y^{n+a-1}(Y_{\text{ét}}, \mu_D) \rightarrow H^{n+a-1}(Y_{\text{ét}}, \mu_D) \rightarrow 0$$

is exact by computing the spectral sequence

$$(4.5) \quad E_1^{u,v} = \bigoplus_{y \in Y^u} H_y^{u+v}(Y_{\text{ét}}, \mu_D) \implies H^{u+v}(Y_{\text{ét}}, \mu_D).$$

Since $\dim(Y) = a - 1$, $E_1^{u,v}$ is zero unless $0 \leq u \leq a - 1$. We have $E_1^{u,v} = 0$ for $v \geq n + 2$ by Lemma 4.2 (2), and $E_2^{u,n+1}$ is zero for $0 \leq u \leq a - 2$ by Lemma 4.2 (3) for X . Hence we obtain $E_2^{a-1,n} \simeq H^{n+a-1}(Y_{\text{ét}}, \mu_D)$ and the exactness in question. \square

Corollary 4.6. *Assume $\mathbf{S}(K, n, \ell)$ holds. Then the map*

$$\text{cl}_{X,D,x,\ell^r}^{n,\text{loc}} : H_x^a(X_{\text{Nis}}, \mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D)) / \ell^r \longrightarrow H_x^{n+a}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes n})$$

is surjective.

Proof. If $a \leq 1$, then $\text{cl}_{X,D,X,\ell^r}^{n,\text{loc}}$ is surjective by its construction in §3 and the assumption $\mathbf{S}(K, n, \ell)$ (see also [1], 5.13 (i)). Here we have used the surjectivity of the lower δ in

(3.3). We prove the case $a \geq 2$ by induction on a . By the induction hypothesis and the construction of $\mathrm{cl}_{X,D,x,\ell^r}^{n,\mathrm{loc}}$ (cf. (3.4)), it is enough to show that the connecting map

$$\bigoplus_{y \in X^{a-1}} H_y^{n+a-1}(X_{\mathrm{\acute{e}t}}, \mu_D) \longrightarrow H_x^{n+a}(X_{\mathrm{\acute{e}t}}, \mu_D)$$

is surjective, which has been shown in Corollary 4.3. \square

In the rest of this section, we prove Lemma 4.2.

Proof of Lemma 4.2. (1) By [22], I.3.3, Proposition 14, it is enough to deal with the case $R = A$. Then the assertion follows from Theorem 4.1 applied to $R' = \mathrm{Im}(A \rightarrow \kappa(y))$ and the assumption that the residue field of A is finite.

(2) follows from (1) and a standard localization argument using the excision in the étale topology. The details are straight-forward and left to the reader.

(3) By a result of Panin [17], 5.2 (cf. [2]), the sequence

$$(4.7) \quad \begin{aligned} 0 \rightarrow H^{n+1}(\eta_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) &\rightarrow \bigoplus_{y \in X^1} H_y^{n+2}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) \rightarrow \bigoplus_{y \in X^2} H_y^{n+3}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) \rightarrow \cdots \\ &\rightarrow H_x^{n+a+1}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) \rightarrow 0 \end{aligned}$$

is exact, where we have used the vanishing $H^{n+1}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) = 0$. Indeed, we have

$$H^q(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) \simeq H^q(x_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) = 0 \quad \text{for } q > n + 1 - a$$

by Lemma 4.2 (1), and we have $n+1 > n+1-a$ by the assumption that $a (= \dim(X)) \geq 1$. Thus we obtain the assertion if $D = 0$. In particular,

$$(4.8) \quad H^{n+1}(\eta_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) \simeq H_x^{n+2}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}), \quad \text{if } a = 1.$$

In what follows, assume $D \neq 0$. We prove the sequence in question is isomorphic to the exact sequence (4.7). It is enough to show the natural map

$$(4.9) \quad H_x^{n+a+1}(X_{\mathrm{\acute{e}t}}, \mu_D) \longrightarrow H_x^{n+a+1}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n})$$

is bijective for any X and any non-zero D on X as in Lemma 4.2, where x denotes the closed point of X and a denotes $\dim(X)$. We prove this assertion by induction on $a = \dim(X)$. If $a = 1$, then the assertion follows from the isomorphisms

$$H_x^{n+2}(X_{\mathrm{\acute{e}t}}, \mu_D) \xleftarrow{\sim} H^{n+1}(\eta_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}) \xrightarrow[(4.8)]{\sim} H_x^{n+2}(X_{\mathrm{\acute{e}t}}, \mu_{\ell^r}^{\otimes n}),$$

where the left arrow is bijective by the fact that $\mu_D|_x = 0$. Assume $a \geq 2$. Computing the spectral sequence (4.5) using Lemma 4.2 (2), one can easily check the sequence

$$\bigoplus_{y \in Y^{a-2}} H_y^{n+a-1}(Y_{\mathrm{\acute{e}t}}, \mu_D) \rightarrow \bigoplus_{y \in Y^{a-1}} H_y^{n+a}(Y_{\mathrm{\acute{e}t}}, \mu_D) \rightarrow H^{n+a}(Y_{\mathrm{\acute{e}t}}, \mu_D) \rightarrow 0$$

is exact. Since $\mu_D|_x = 0$, we have $H^{n+a}(Y_{\mathrm{\acute{e}t}}, \mu_D) \simeq H_x^{n+a+1}(X_{\mathrm{\acute{e}t}}, \mu_D)$. Thus the sequence

$$\bigoplus_{y \in X^{a-2}} H_y^{n+a-1}(X_{\mathrm{\acute{e}t}}, \mu_D) \rightarrow \bigoplus_{y \in X^{a-1}} H_y^{n+a}(X_{\mathrm{\acute{e}t}}, \mu_D) \rightarrow H_x^{n+a+1}(X_{\mathrm{\acute{e}t}}, \mu_D) \rightarrow 0$$

is exact. We obtain the bijectivity of (4.9) by comparing this exact sequence with (4.7) using the induction hypothesis. This completes the proof of the lemma. \square

5. PROOF OF THEOREM 1.2

We introduce some auxiliary terminology and notation. For $0 \leq a \leq n-1$, we define a chain over A of length a to be a sequence

$$(5.1) \quad (\mathfrak{m}_A, \mathfrak{p}_{n-1}^n, \mathfrak{p}_{n-2}^{n-1}, \mathfrak{p}_{n-3}^{n-2}, \dots, \mathfrak{p}_{n-a}^{n-a+1}),$$

where \mathfrak{m}_A is the maximal ideal of A and \mathfrak{p}_{n-1}^n is a prime ideal of A of height $n-1$. For $2 \leq q \leq a$, $\mathfrak{p}_{n-q}^{n-q+1}$ is a prime ideal of height $n-q$ of the $(n-q+1)$ -dimensional henselian local ring

$$\left(\dots \left(\left(A_{\mathfrak{p}_{n-1}^n}^h \right)_{\mathfrak{p}_{n-2}^{n-1}}^h \right) \dots \right)_{\mathfrak{p}_{n-q+1}^{n-q+2}}^h.$$

For a chain δ over A of length a of the form (5.1), we define the henselian local ring A_δ^h of dimension $n-a$ as

$$A_\delta^h := \left(\dots \left(\left(A_{\mathfrak{p}_{n-1}^n}^h \right)_{\mathfrak{p}_{n-2}^{n-1}}^h \right) \dots \right)_{\mathfrak{p}_{n-a}^{n-a+1}}^h.$$

For δ of length zero, A_δ^h means A itself. We prove

Proposition 5.2. *Assume $1 \leq a \leq n$ and that $\mathbf{S}(K, n, \ell)$ holds. Let δ be a chain over A of length $n-a$, and let D be an effective divisor on $X := \text{Spec}(A_\delta^h)$. Let j be the open immersion $X - \text{Supp}(D) \hookrightarrow X$ and let x be the closed point of X . Then the map*

$$\text{cl}_{X,D,x,\ell^r}^{n,\text{loc}} : H_x^a(X_{\text{Nis}}, \mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}_D)) / \ell^r \longrightarrow H_x^{n+a}(X_{\text{ét}}, j_! \mu_{\ell^r}^{\otimes n})$$

is bijective.

Theorem 1.2 follows from the case $a = n$ of Proposition 5.2.

Proof. Assume $a = 1$. The generic point η (resp. the closed point x) of X is the spectrum of an n -dimensional local field (resp. $(n-1)$ -dimensional local field) in the sense of [9], and the Galois symbol maps of η and x are bijective by Remark 1.5 (5). Hence $\text{cl}_{X,D,x,\ell^r}^{n,\text{loc}}$ is bijective by the diagram (3.3) and Lemma 2.1 (1). The case $a \geq 2$ follows from the induction on a using the diagram (3.4) and Corollaries 4.3 and 4.6. \square

This completes the proof of Theorem 1.2.

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